

RADIANT HEAT TRANSFER WITH SCATTERING

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Abstract—The transport equation appropriate for radiant heat-transfer calculations and including the effects of photon scattering is discussed. It is shown that for low temperatures and photon energies ($\ll 0.5$ MeV) the scattering of photons from a Maxwellian gas of free electrons can be described by a relatively simple second order differential operator. A diffusion approximation, employing the same description of scattering, is derived from the transport equation. The qualitative aspects of the solution of both the transport and diffusion equations are discussed in the limit of zero electron temperature. In particular, it is shown that in this limit this description of scattering gives the proper behavior of only a decrease in photon frequency upon scattering. This is in contrast to previous attempts to represent the scattering process by a differential operator which led to the physically incorrect result of some increase in frequency due to scattering from electrons at rest.

NOMENCLATURE

B ,	source due to spontaneous emission;	μ'_a ,	absorption coefficient modified to include induced emission;
c ,	speed of light;	μ_s ,	Thomson scattering coefficient;
D ,	diffusion coefficient;	ν ,	frequency;
e ,	electronic charge;	σ_s ,	double differential scattering coefficient;
h ,	Planck's constant;	τ_e ,	hydrodynamic time scale;
I ,	specific intensity;	τ_r ,	radiation time scale;
I_0 ,	zeroth angular moment of specific intensity;	Ω ,	unit vector in direction of photon travel.
I_1 ,	first angular moment of specific intensity;		
k ,	Boltzmann's constant;		
m ,	electron mass;		
\mathbf{n} ,	unit outward normal vector;		
N_e ,	electron density;		
\mathbf{r} ,	spatial coordinate;		
r_0 ,	classical electron radius;		
t ,	time;		
T ,	temperature;		
x ,	dimensionless wavelength.		

Greek symbols	
α ,	dimensionless temperature;
γ ,	dimensionless frequency;
Γ ,	incoming intensity distribution;
Λ ,	initial intensity distribution;
μ_a ,	absorption coefficient;

1. INTRODUCTION

IN THIS paper a relatively simple treatment of photon scattering in the transport equation appropriate for radiant heat-transfer calculations is presented. In particular the description of photon scattering from a Maxwellian gas of free electrons is considered in detail within the radiative transfer context. The most comprehensive treatment of this problem to date is due to Fraser [1] and we shall draw heavily upon his ideas. However, it will be shown that Fraser's analysis can be carried further and the resulting description of scattering is both simpler and qualitatively more accurate than Fraser's result. Following a brief review of work done to date on this scattering problem, we shall be able to state more explicitly the contribution of this paper.

In the polarization independent approximation the equation of transfer for the specific intensity $I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)$ of photons can be written quite generally as [2]

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) \\ &= \mu'_a [B(\nu) - I(\nu, \boldsymbol{\Omega})] \\ & - \int_0^\infty d\nu' \int_{4\pi} d\boldsymbol{\Omega}' \sigma_s(\nu \rightarrow \nu', \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') I(\nu', \boldsymbol{\Omega}') \\ & \times \left[1 + \frac{c^2}{2h\nu'^3} I(\nu', \boldsymbol{\Omega}') \right] \\ & + \int_0^\infty d\nu' \int_{4\pi} d\boldsymbol{\Omega}' \sigma_s(\nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) I(\nu', \boldsymbol{\Omega}') \\ & \times \frac{\nu}{\nu'} \left[1 + \frac{c^2}{2h\nu^3} I(\nu, \boldsymbol{\Omega}) \right], \quad (1) \end{aligned}$$

where, for notational simplicity, we have dropped the arguments \mathbf{r} and t in the specific intensity $I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)$ on the right-hand side of this equation. Here \mathbf{r} , ν , $\boldsymbol{\Omega}$, and t are the spatial, frequency, angular, and temporal coordinates, respectively, c is the speed of light, $B(\nu)$ is the source term due to spontaneous emission, $\mu'_a(\nu)$ is the absorption coefficient suitably modified for induced emission, and $\sigma_s(\nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})$ is the double differential scattering coefficient (cross section). Although not explicitly shown, B , μ'_a , and σ_s are in general functions of both \mathbf{r} and t . We assume, however, that $B(\nu)$ and $\mu'_a(\nu)$ are independent of the direction $\boldsymbol{\Omega}$ of travel of the photon, and, as shown, that $\sigma_s(\nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})$ is only a function of the scattering angle rather than $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ separately. For an observer at rest with respect to the medium this implies isotropic matter, i.e. matter with no preferential direction. For an observer moving with speed v with respect to the medium, this also implies the neglect of terms of order v/c [1]. If local thermodynamic equilibrium can be

assumed, then $B(\nu)$ is the Planck function

$$B(\nu) = \frac{2h\nu^3}{c^2} (e^{h\nu/kT} - 1)^{-1}, \quad (2)$$

and

$$\mu'_a = \mu_a(1 - e^{-h\nu/kT}), \quad (3)$$

where μ_a is the absorption coefficient appropriate to thermodynamic equilibrium and the exponential factor is the effective decrease in absorption due to stimulated emission. Here $T = T(\mathbf{r}, t)$ is the local temperature of the medium. Stimulated scattering is described by the quadratic terms in the intensity in equation (1). The factor ν/ν' in the inscattering term accounts for the fact that the scattering coefficient is defined relative to a photon density in phase space whereas the intensity I is an energy density. The only significant approximation we have made in writing equation (1), other than the inherent approximation that photon transport can be described by a classical equation of transfer, is that polarization effects need not be taken into account. If the scattering interaction is between photons and free electrons, analytical evidence suggests that, as far as energy transfer is concerned, this approximation introduces a very small error [3]. Of course, if one is specifically interested in polarization effects, equation (1) is not applicable.

In the case of photon scattering from free electrons at rest, the scattering interaction is described by the well known Klein-Nishina formula [4]

$$\begin{aligned} \sigma_s(\nu \rightarrow \nu', \mu) &= N_e r_0^2 \left(\frac{1 + \mu^2}{2} \right) [1 + \gamma(1 - \mu)]^{-2} \\ & \times \left\{ 1 + \frac{\gamma^2(1 - \mu)^2}{(1 + \mu^2)[1 + \gamma(1 - \mu)]} \right\} \\ & \times \delta \left(\nu' - \frac{\nu}{1 + \gamma(1 - \mu)} \right), \quad (4) \end{aligned}$$

where N_e is the electron density, $\gamma = h\nu/mc^2$ (h = Planck's constant and mc^2 is the rest energy of the electron), $r_0 = e^2/mc^2$ is the classical electron radius, and $\delta(z)$ is the Dirac

delta function. The case of scattering from a moving electron can be treated by using equation (4) in conjunction with a Lorentz transformation of equation (1). Finally, the case of scattering from a distribution of moving electrons can be treated by averaging the Lorentz transformed equation for a particular velocity over the velocity distribution of the electrons. This procedure leads to an expression for the scattering coefficient $\sigma_s(v \rightarrow v', \mu)$ and, for a Maxwellian electron distribution, was initially carried out by Dirac [5] and later refined by Edmonds [6]. However, the result is algebraically complex and can only be expressed relatively simply as a power series in $(kT/mc^2)^\ddagger$, where T is the electron temperature (k is Boltzmann's constant).

Because of this complexity, one is led to seek a simpler description of photon scattering from electrons in the equation of radiative transfer. The first step in this direction was taken by Chandrasekhar [7] who treated, within the context of a specific problem, the case of scattering from electrons at rest. The essential idea of Chandrasekhar was to expand $I(\mathbf{r}, v', \Omega', t)$ in the scattering terms of equation (1) in a power series about $v' = v$ and carry only zeroth and first derivative terms. This procedure, which is correct to order $\gamma = hv/mc^2$, transforms the integral operator in frequency into a first order differential operator and is equivalent to the "Fermi Age" approximation in neutron transport theory [8]. However, the work of Chandrasekhar lacked generality since it treated the case of zero electron temperature and dilute radiation (the non-linear induced scattering terms were neglected). Further, an approximation to the Klein-Nishina formula given by

$$\sigma_s(v \rightarrow v', \mu) = \frac{2}{3} N_e r_0^2 \delta \left(v' - \frac{v}{1 + \gamma(1 - \mu)} \right), \quad (5)$$

was used in this work. In addition to Chandrasekhar, several other authors [9-12] have

used this procedure, or slight modifications of it, to compute changes in the spectral distribution of radiation due to scattering from electrons at rest. All have found a physically incorrect behavior, namely a slight increase in frequency for some photons upon scattering. This error is clearly introduced by making use of the truncated Taylor series expansion since the more correct integral operator will not display this incorrect behavior.

This procedure of Chandrasekhar has recently been generalized by Fraser [1] to include both the effects of a non-zero electron temperature and induced scattering effects. Further, Fraser used the correct Klein-Nishina scattering coefficient, equation (4), as the rest frame description. Fraser's result is an equation of transfer correct to order $\gamma = hv/mc^2$ and $\alpha = kT/mc^2$ with scattering represented by a second order differential operator. Since $mc^2 \approx 0.51$ meV, this equation should be quite accurate for most problems of interest. However, Fraser's formulation still suffers from the defect that in the limit of zero electron temperature it also gives the incorrect result of some increase in photon frequency upon scattering [13].

With this background discussion we can now state explicitly the content of the present paper. We shall show that Fraser's equation of transfer can be simplified without invoking any assumptions other than those already contained in his result. Further, we shall discuss the available analytic evidence which indicates that in the limit of zero electron temperature this simplified equation of transfer does not give the incorrect behavior of a frequency increase upon scattering. We shall prove this result in some degree of generality without reference to any specific problem. We will also derive the $P - 1$ (diffusion) approximation, equivalent to the two point Gauss quadrature method often used in analytic work in planar and spherical geometries [14], to the simplified transport equation. We shall show that this equation also gives the proper behavior in the limit of zero electron temperature.

2. EQUATION OF TRANSFER

Fraser [1] has shown that an expansion of equation (1) to first order in $\gamma = hv/mc^2$ and $\alpha = kT/mc^2$ in the case of photon scattering from a Maxwellian gas of free electrons at temperature T yields

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) \\ &= \mu'_a [B(\nu) - I(\nu, \boldsymbol{\Omega})] - \mu_s (1 - 2\gamma) I(\nu, \boldsymbol{\Omega}) \\ &+ \mu_s \int_{4\pi} d\boldsymbol{\Omega}' \sum_{n=0}^3 \left(\frac{2n+1}{4\pi} \right) \\ &\times P_n(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') S_n I(\nu, \boldsymbol{\Omega}') \\ &- \frac{3\mu_s}{16\pi} \frac{c^2}{h\nu^3} \gamma I(\nu, \boldsymbol{\Omega}) \left(1 - \nu \frac{\partial}{\partial \nu} \right) \int_{4\pi} \\ &\times d\boldsymbol{\Omega}' [1 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2 \\ &\quad - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^3] I(\nu, \boldsymbol{\Omega}'), \quad (6) \end{aligned}$$

where $\mu_s = 8\pi N_e r_0^2/3$ is the Thomson scattering coefficient, $P_n(z)$ is the n th Legendre polynomial, and the operators S_n are defined as

$$\begin{aligned} S_0 = & \left[1 - \gamma \left(1 - \nu \frac{\partial}{\partial \nu} \right) \right. \\ & \left. - \alpha \left(2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (7) \end{aligned}$$

$$\begin{aligned} S_1 = & \frac{2}{5} \left[\gamma \left(1 - \nu \frac{\partial}{\partial \nu} \right) \right. \\ & \left. - \alpha \left(1 - 2\nu \frac{\partial}{\partial \nu} + \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (8) \end{aligned}$$

$$\begin{aligned} S_2 = & \frac{1}{10} \left[1 - \gamma \left(1 - \nu \frac{\partial}{\partial \nu} \right) \right. \\ & \left. - \alpha \left(6 + 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (9) \end{aligned}$$

$$\begin{aligned} S_3 = & \frac{3}{70} \left[\gamma \left(1 - \nu \frac{\partial}{\partial \nu} \right) \right. \\ & \left. + \alpha \left(4 + 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right]. \quad (10) \end{aligned}$$

Since equation (6) is the lowest order description,

in α and γ , of deviations from conservative Thomson scattering, it can be simplified without introducing any further approximations.

A straightforward way to effect this simplification is to consider equation (6) projected onto the basis elements of a spherical harmonic function space. We shall follow the vectorial method introduced in neutron transport theory. Since these spherical harmonic equations are only an intermediate result, we shall omit the details of their derivation, referring the interested reader to the book by Davison [15]. If for simplicity we momentarily neglect the induced scattering (non-linear) terms in equation (6), the result is

$$\begin{aligned} & \frac{1}{c} \frac{\partial J_0}{\partial t} + \nabla_U \cdot \nabla_r J_1 + \mu'_a (J_0 - 4\pi B) \\ & \quad + \mu_s (1 - 2\gamma - S_0) J_0 = 0, \quad (11) \end{aligned}$$

$$\begin{aligned} & \frac{3}{c} \frac{\partial J_1}{\partial t} + \nabla_U \cdot \nabla_r J_2 + 3[\mu'_a + \mu_s (1 - 2\gamma \\ & \quad - S_1)] J_1 + \mathbf{U} \cdot \nabla_r J_0 = 0, \quad (12) \end{aligned}$$

$$\begin{aligned} & \frac{5}{c} \frac{\partial J_2}{\partial t} + \nabla_U \cdot \nabla_r J_3 + 5[\mu'_a + \mu_s (1 - 2\gamma \\ & \quad - S_2)] J_2 + 3\mathbf{U} \cdot \nabla_r J_1 - \mathbf{U}^2 \nabla_U \cdot \nabla_r J_1 = 0, \quad (13) \end{aligned}$$

$$\begin{aligned} & \frac{7}{c} \frac{\partial J_3}{\partial t} + \nabla_U \cdot \nabla_r J_4 + 7[\mu'_a + \mu_s (1 - 2\gamma \\ & \quad - S_3)] J_3 + 5\mathbf{U} \cdot \nabla_r J_2 - \mathbf{U}^2 \nabla_U \cdot \nabla_r J_2 = 0, \quad (14) \end{aligned}$$

$$\begin{aligned} & \frac{2n+1}{c} \frac{\partial J_n}{\partial t} + \nabla_U \cdot \nabla_r J_{n+1} + (2n+1) \\ & \times [\mu'_a + \mu_s (1 - 2\gamma)] J_n + (2n-1) \mathbf{U} \cdot \nabla_r J_{n-1} \\ & \quad - \mathbf{U}^2 \nabla_U \cdot \nabla_r J_{n-1} = 0, n \geq 4. \quad (15) \end{aligned}$$

In these equations the vector \mathbf{U} is in the direction $\boldsymbol{\Omega}$ and has an arbitrary magnitude U . The functions J_n are defined as

$$J_n = \frac{U^n}{2n+1} \sum_{m=-n}^n A_{nm} I_{nm}(\mathbf{r}, \nu, t) Y_{nm}(\boldsymbol{\Omega}), \quad (16)$$

where the $I_{nm}(\mathbf{r}, \nu, t)$ are the coefficients of an expansion of the specific intensity in surface

harmonics according to

$$I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm} I_{nm}(\mathbf{r}, \nu, t) Y_{nm}(\boldsymbol{\Omega}). \quad (17)$$

Here the surface harmonics are defined in the usual way:

$$Y_{nm}(\boldsymbol{\Omega}) = P_n^{|m|}(\cos \theta) e^{im\phi}, \quad (18)$$

where the $P_n^m(z)$ are the associated Legendre functions and the constants A_{nm} are normalization coefficients

$$\begin{aligned} A_{nm} &= 4\pi \left[\int_{4\pi} d\boldsymbol{\Omega} Y_{nm}(\boldsymbol{\Omega}) Y_{nm}^*(\boldsymbol{\Omega}) \right]^{-1} \\ &= \frac{(2n+1)(n-|m|)!}{(n+|m|)!} \end{aligned} \quad (19)$$

with the asterisk on $Y_{nm}(\boldsymbol{\Omega})$ indicating the complex conjugate. Due to the biorthogonality relationship between the surface harmonics and their complex conjugates, one has an explicit expression for $I_{nm}(\mathbf{r}, \nu, t)$ in terms of the specific intensity, i.e.

$$I_{nm}(\mathbf{r}, \nu, t) = \int_{4\pi} d\boldsymbol{\Omega} Y_{nm}^*(\boldsymbol{\Omega}) I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t). \quad (20)$$

Since J_n consists of $2n + 1$ angular components [see equation (16)], equation (15) represents in general $2n + 1$ relationships between the components of J_{n-1} , J_n , and J_{n+1} . In special cases symmetry considerations may reduce the number of non-zero components of J_n , and the number of relationships in equation (15) is reduced accordingly. For example, in plane parallel problems the specific intensity is independent of the azimuthal angle ϕ , and hence J_n consists of only one non-zero component.

Now, in equation (12) we replace $\mu_s(1 - 2\gamma - S_1)$ by just μ_s , since $S_1 + 2\gamma$ is of order α and γ and hence to lowest order $\mu_s(1 - 2\gamma - S_1) = \mu_s$. By similar arguments, we replace $\mu_s(1 - 2\gamma - S_2)$ in equation (13) by $9\mu_s/10$, and in equations (14) and (15) we replace $\mu_s(1 - 2\gamma - S_3)$ and $\mu_s(1 - 2\gamma)$ in each case by μ_s . We note, however, that we cannot make a

similar simplification in equation (11) since $(1 - 2\gamma - S_0)$ is of order α and γ , rather than of order unity (or $9/10$), as are the similar terms in equations (12–15). Introducing these simplifications into equations (12–15) we find that equations (11–15) are the spherical harmonic projections of the equation of transfer

$$\begin{aligned} &\frac{1}{c} \frac{\partial I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) \\ &= \mu'_a [B(\nu) - I(\nu, \boldsymbol{\Omega})] - \mu_s I(\nu, \boldsymbol{\Omega}) \\ &\quad + \frac{3\mu_s}{16\pi} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})^2] I(\nu, \boldsymbol{\Omega}') \\ &\quad + \frac{\mu_s}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \left[\alpha \nu^2 \frac{\partial^2}{\partial \nu^2} + (\gamma - 2\alpha)\nu \frac{\partial}{\partial \nu} + \gamma \right] \\ &\quad \times I(\nu, \boldsymbol{\Omega}'). \end{aligned} \quad (21)$$

To equation (21) we need add the contribution of the non-linear induced scattering terms in equation (6). Since these terms are of order γ , they can be neglected in all but the zeroth angular moment of the equation of transfer, just as we neglected all terms of order α and γ in the linear analysis just completed except in the zeroth angular moment relationship, equation (11). This implies the replacement in the equation of transfer:

$$\begin{aligned} &I(\nu, \boldsymbol{\Omega}) [1 - \nu(\partial/\partial \nu)] \int_{4\pi} d\boldsymbol{\Omega}' [1 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')] \\ &\quad + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^3] I(\nu, \boldsymbol{\Omega}') \\ &\rightarrow (1/4\pi) \int_{4\pi} d\boldsymbol{\Omega} I(\nu, \boldsymbol{\Omega}) \int_{4\pi} d\boldsymbol{\Omega}' [1 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')] \\ &\quad + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2 - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^3] [1 - \nu(\partial/\partial \nu)] I(\nu, \boldsymbol{\Omega}'). \end{aligned} \quad (22)$$

Thus, the full form of equation (21), including the

effects of induced scattering, is

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) \\
&= \mu'_a [B(\nu) - I(\nu, \boldsymbol{\Omega})] - \mu_s I(\nu, \boldsymbol{\Omega}) \\
&+ \frac{3\mu_s}{16\pi} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] I(\nu, \boldsymbol{\Omega}') \\
&+ \frac{\mu_s}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \left[\alpha \nu^2 \frac{\partial^2}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial}{\partial \nu} + \gamma \right] \\
&\times I(\nu, \boldsymbol{\Omega}') - \frac{3\mu_s}{64\pi^2} \frac{c^2}{h\nu^3} \gamma \int_{4\pi} d\boldsymbol{\Omega}'' I(\nu, \boldsymbol{\Omega}'') \int_{4\pi} d\boldsymbol{\Omega}''' \\
&\times [1 - (\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}'') + (\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}''')^2 - (\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}''')^3] \\
&\quad \times \left(1 - \nu \frac{\partial}{\partial \nu} \right) I(\nu, \boldsymbol{\Omega}'''), \quad (23)
\end{aligned}$$

which is a simplified, but *a priori* just as accurate, form of Fraser's result, equation (6). In particular, equation (23) contains far less scattering terms than does equation (6), and the terms which account for energy transfer in the scattering interaction, i.e. those proportional to α and γ , are isotropic in equation (23) whereas they are angularly dependent in equation (6). Both of these facts should make equation (23) much easier to solve, either analytically or numerically, than equation (6). A significant property of equation (23) is that it gives the proper equilibrium solution, namely a Planck distribution at temperature T as given by equation (2). This can be verified by direct substitution. Equation (23) needs to be supplemented with the usual transport initial condition

$$I(\mathbf{r}, \nu, \boldsymbol{\Omega}, 0) = A(\mathbf{r}, \nu, \boldsymbol{\Omega}), \quad (24)$$

and boundary condition which, for a convex body, is

$$I(\mathbf{r}_s, \nu, \boldsymbol{\Omega}, t) = \Gamma(\mathbf{r}_s, \nu, \boldsymbol{\Omega}, t), \quad \boldsymbol{\Omega} \cdot \mathbf{n} < 0, \quad (25)$$

where A and Γ are specified (given) functions, \mathbf{r}_s denotes a point on the boundary of the system under consideration, and \mathbf{n} is a unit outward normal vector.

In the final section of this paper we shall

discuss the accuracy of equation (23) in the limit of zero electron temperature ($\alpha = 0$). Before doing this, however, we derive the diffusion approximation to this transport equation.

3. THE DIFFUSION APPROXIMATION

In problems involving planar or spherical symmetry, a commonly used approximation in treating any transport-like equation either analytically or numerically is the representation of integrals over $\boldsymbol{\Omega}$ by a two point Gauss quadrature formula [14]. This approximation is equivalent to expanding the intensity in Legendre polynomials and carrying only the first two terms. This latter viewpoint has the advantage that it generalizes in a straightforward way to a general geometric situation. That is, the specific intensity is represented by the first two terms in a surface harmonic expansion

$$I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) = \frac{1}{4\pi} I_0(\mathbf{r}, \nu, t) + \frac{3}{4\pi} \boldsymbol{\Omega} \cdot \mathbf{I}_1(\mathbf{r}, \nu, t), \quad (26)$$

where the expansion coefficients I_0 and \mathbf{I}_1 are related to the energy density E and net flux \mathbf{F} per unit frequency according to

$$E(\mathbf{r}, \nu, t) \equiv \frac{1}{c} \int_{4\pi} d\boldsymbol{\Omega} I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) = \frac{1}{c} I_0(\mathbf{r}, \nu, t), \quad (27)$$

$$\mathbf{F}(\mathbf{r}, \nu, t) \equiv \int_{4\pi} d\boldsymbol{\Omega} \boldsymbol{\Omega} I(\mathbf{r}, \nu, \boldsymbol{\Omega}, t) = \mathbf{I}_1(\mathbf{r}, \nu, t). \quad (28)$$

We now use the assumed representation, equation (26), in the simplified equation of transfer, equation (23), multiply the result by 1, $\boldsymbol{\Omega}_x$, $\boldsymbol{\Omega}_y$, and $\boldsymbol{\Omega}_z$, respectively, and integrate over all solid angle. Suppressing the algebra, we find that the four moment equations, generally called the P -1 equations, can be written

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I_0(\mathbf{r}, \nu, t)}{\partial t} + \nabla \cdot \mathbf{I}_1(\mathbf{r}, \nu, t) = \mu'_a [4\pi B(\nu) - I_0(\nu)] \\
&+ \mu_s \left[\alpha \nu^2 \frac{\partial^2 I_0}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial I_0}{\partial \nu} + \gamma I_0 \right] \\
&- \frac{\mu_s c^2 \gamma}{4\pi h \nu^3} \left[I_0^2 - \nu I_0 \frac{\partial I_0}{\partial \nu} - \frac{6}{5} \mathbf{I}_1 \cdot \mathbf{I}_1 + \frac{6}{5} \nu \mathbf{I}_1 \cdot \frac{\partial \mathbf{I}_1}{\partial \nu} \right], \quad (29)
\end{aligned}$$

$$\frac{1}{c} \frac{\partial \mathbf{I}_1(\mathbf{r}, \nu, t)}{\partial t} + (1/3) \nabla I_0(\mathbf{r}, \nu, t) + (\mu'_a + \mu_s) \mathbf{I}_1(\nu) = 0. \quad (30)$$

To obtain what we shall call the diffusion approximation, two additional simplifications are required. In equation (29) we need neglect the $\mathbf{I}_1 \cdot \mathbf{I}_1$ terms compared to the I_0^2 terms. This neglect is consistent with the two term surface harmonic representation, equation (26), since this representation is only strictly correct if $|\mathbf{I}_1| \ll I_0$. Secondly, we need neglect the $\partial \mathbf{I}_1 / \partial t$ term in equation (30) which, if carried, would give rise to a wave rather than a pure diffusion character. Making these two simplifications and eliminating \mathbf{I}_1 between the resulting two equations, we find as the diffusion approximation

$$\begin{aligned} \frac{1}{c} \frac{\partial I_0(\mathbf{r}, \nu, t)}{\partial t} - \nabla \cdot D \nabla I_0(\mathbf{r}, \nu, t) = \mu'_a [4\pi B - I_0(\nu)] \\ + \mu_s \left[\alpha \nu^2 \frac{\partial^2 I_0}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial I_0}{\partial \nu} + \gamma I_0 \right. \\ \left. + \frac{c^2 \nu}{4\pi h \nu^3} I_0 \left(\nu \frac{\partial I_0}{\partial \nu} - I_0 \right) \right], \quad (31) \end{aligned}$$

where D , the diffusion coefficient, is defined as

$$D = \frac{1}{3(\mu'_a + \mu_s)}. \quad (32)$$

Equation (31) has been obtained earlier [3] in a somewhat different manner. Although much simpler than the transport equation (23), the diffusion equation (31) should contain all of the significant physics, in a semi-quantitative sense, of the radiative energy transport problem including scattering effects. The initial condition, equation (24), becomes for the diffusion equation

$$I_0(\mathbf{r}, \nu, 0) = \int_{4\pi} d\Omega A(\mathbf{r}, \nu, \Omega) \equiv A(\mathbf{r}, \nu), \quad (33)$$

and the Marshak boundary condition, an approximation to the transport boundary con-

dition, equation (25), is [15, 16]

$$\begin{aligned} \int_{\mathbf{n} \cdot \boldsymbol{\Omega} < 0} d\Omega |\mathbf{n} \cdot \boldsymbol{\Omega}| \Gamma(\mathbf{r}_s, \nu, \boldsymbol{\Omega}, t) \equiv \Gamma(\mathbf{r}_s, \nu, t) \\ = (1/4) I_0(\mathbf{r}_s, \nu, t) + (1/2) D \mathbf{n} \cdot \nabla I_0(\mathbf{r}_s, \nu, t). \quad (34) \end{aligned}$$

4. THE ZERO TEMPERATURE LIMIT

It was pointed out in the introduction that previous work which represented the scattering process by a differential operator led in all cases to a transport equation which predicted the physically incorrect result of some increase in the photon frequency as a result of scattering from electrons at rest. The particular problem treated by these authors [7, 9–12] in which this behaviour manifested itself was that of the steady-state transmission of radiation through a finite atmosphere. Very recently, this same problem was treated [13] using the diffusion approximation given in this paper, equation (31), as the description of radiative transfer. In this case the correct behaviour of only a decrease in photon frequency due to scattering was observed. It would be a significant finding if one could prove in general, without reference to any particular problem, that the diffusion approximation, equation (31), as well as the more rigorous transport description, equation (23), always displays this qualitatively correct behaviour. We believe this to be the case and give in this section our progress to date in constructing such a proof. The analysis we shall give encompasses a large class of physical situations, but we shall have to appeal to physical arguments to extend the validity of our proof to the most general case. It is hoped that in the future a proof can be constructed which does not require these physical arguments.

We first consider the diffusion approximation, equations (31, 33, 34). We imagine that the source function $B(\nu)$, the initial condition $A(\nu)$, and the incoming distribution $\Gamma(\nu)$ vanish for frequencies greater than some frequency ν_0 . We need prove that the resulting solution for $I_0(\mathbf{r}, \nu, t)$ vanishes for $\nu > \nu_0$ in the case of zero

electron temperature. It is convenient to introduce a new independent variable x

$$x - x_0 = 1/\gamma = mc^2/hv, \quad (35)$$

where $x_0 = 1/\gamma_0 = mc^2/hv_0$, and a new dependent variable \tilde{I}_0

$$\tilde{I}_0(\mathbf{r}, x, t) = I_0(\mathbf{r}, v, t)/(x + x_0). \quad (36)$$

If we further define

$$\tilde{B}(\mathbf{r}, x, t) = B(\mathbf{r}, v, t)/(x + x_0), \quad (37)$$

$$\tilde{A}(\mathbf{r}, x) = A(\mathbf{r}, v)/(x + x_0), \quad (38)$$

$$\tilde{\Gamma}(\mathbf{r}_s, x, t) = \Gamma(\mathbf{r}_s, v, t)/(x + x_0), \quad (39)$$

then in the zero electron temperature ($\alpha = 0$) case the diffusion approximation, equations (31, 33, 34) become, with all tildas dropped,

$$\begin{aligned} & \frac{1}{c} \frac{\partial I_0(\mathbf{r}, x, t)}{\partial t} - \mathbf{V} \cdot D \nabla I_0(\mathbf{r}, x, t) \\ & = \mu'_a [4\pi B H(x) - I_0] \\ & - \mu_s \left\{ \frac{\partial I_0}{\partial x} + \beta \frac{\partial}{\partial x} [(x + x_0)^4 I_0^2] \right\}, \\ & \quad -x_0 < x < \infty, \end{aligned} \quad (40)$$

$$I_0(\mathbf{r}, x, 0) = A(\mathbf{r}, x) H(x), \quad (41)$$

$$\begin{aligned} \Gamma(\mathbf{r}_s, x, t) H(x) = \frac{1}{4} I_0(\mathbf{r}_s, x, t) \\ + \frac{1}{2} D \mathbf{n} \cdot \nabla I_0(\mathbf{r}_s, x, t), \end{aligned} \quad (42)$$

where we have defined

$$\beta = \frac{c^2 h^2}{8\pi (mc^2)^3}. \quad (43)$$

In these equations $H(x)$ is the unit step (Heaviside) function which explicitly indicates that the source, initial condition, and incoming distribution vanish for negative x ($x < 0$ corresponds to $v > v_0$). In terms of these new variables we need prove that $I_0(\mathbf{r}, x, t)$ as defined by equations (40–42) vanishes for negative x .

We treat here the linear diffusion equation, i.e. equation (40) without the induced scattering terms ($\beta = 0$). However, the proof we shall give is applicable in the general case $\beta \neq 0$ if the non-linear terms can be treated by a perturbation

method. This method consists of initially neglecting the non-linear terms and solving the linear equation for I_0 , using this solution to compute the non-linear terms, solving the resulting equation which is again linear for I_0 , etc. With the assumption that this procedure converges to the correct solution for $x < 0$, the linear analysis we shall give actually treats the non-linear equation. It should be emphasized that this question of convergence has not been investigated and hence the applicability of our conclusions to the non-linear equation is not assured. The analysis we shall give assumes that μ_s and μ'_a are functions of the spatial variable only. On physical grounds, however, we can extend the region of validity of our result in the following way. In the general case $\mu'_a = \mu'_a(\mathbf{r}, x, t)$. In the diffusion equation (40) we retain the full functional dependence of μ'_a in the term $\mu'_a B$, but replace μ'_a in the diffusion coefficient and in the term $\mu'_a I_0$ by its minimum value over the x and t variables. On physical grounds we know that reducing the absorption coefficient in this manner cannot decrease the solution for $I_0(\mathbf{r}, x, t)$. Hence it suffices to show that the intensity vanishes for negative x for a frequency and time independent absorption coefficient. We also consider the solution of the diffusion equation in a time interval Δt such that

$$\tau_r \ll \Delta t \ll \tau_e, \quad (44)$$

where τ_r is the collision time for photons and is given by

$$\tau_r = 1/c(\mu'_a + \mu_s), \quad (45)$$

and τ_e is a characteristic time for the electron density to change. On physical grounds we know such a Δt exists in general since τ_e , a hydrodynamic time scale, is generally much greater than τ_r , a radiation time scale. Now, in the general case $\mu_s = \mu_s(\mathbf{r}, t)$, with the dependence on t (and \mathbf{r}) arising entirely from the time (and space) dependence of the electron density. Since we have chosen $\Delta t \ll \tau_e$, we can assume μ_s to be time independent in this time interval.

On the other hand, since $\Delta t \gg \tau_r$, the radiation intensity will, in general, undergo significant temporal changes in this time interval.

The equation we shall consider is then

$$\begin{aligned} \frac{1}{c} \frac{\partial I_0(\mathbf{r}, x, t)}{\partial t} - \nabla \cdot D \nabla I_0(\mathbf{r}, x, t) + \mu'_a I_0(\mathbf{r}, x, t) \\ = S(\mathbf{r}, x, t) H(x) - \mu_s \frac{\partial I_0(\mathbf{r}, x, t)}{\partial x}, \\ -\infty < x < \infty, \end{aligned} \quad (46)$$

where D , μ'_a , and μ_s are functions of the spatial variable only and we have defined

$$S(\mathbf{r}, x, t) = \mu'_a(\mathbf{r}, x, t) B(\mathbf{r}, x, t). \quad (47)$$

In writing equation (46) we have extended the region of validity of the diffusion equation to include all values of x in the interval $-\infty < x < \infty$. This implies we define $I_0(\mathbf{r}, x, t)$ in the non-physical wavelength region $x < -x_0$ to be the solution of equations (41, 42, 46) which joins continuously at $x = -x_0$ to the physically meaningful solution for $x > -x_0$. We shall use transform methods to show that the intensity as defined by equations (41, 42, 46) vanishes identically for negative x .

We define the Laplace transform of a function of time as

$$\bar{f}(p) = \int_0^{\infty} dt e^{-pt} f(t), \quad \text{Re } p > 0, \quad (48)$$

and the Fourier transform of a function of x as

$$\hat{g}(k) = \int_{-\infty}^{\infty} dx e^{ikx} g(x). \quad (49)$$

In this double transform space equations (41, 42, 46) become

$$\begin{aligned} \frac{p}{c} \hat{I}_0(\mathbf{r}, k, p) - \nabla \cdot D \nabla \hat{I}_0(\mathbf{r}, k, p) \\ + \mu'_a \hat{I}_0(\mathbf{r}, k, p) = \hat{T}(\mathbf{r}, k, p) \\ + ik\mu_s \hat{I}_0(\mathbf{r}, k, p), \end{aligned} \quad (50)$$

$$\begin{aligned} \hat{\Gamma}(\mathbf{r}_s, k, p) = \frac{1}{2} \hat{J}_0(\mathbf{r}_s, k, p) \\ + \frac{1}{2} D \mathbf{n} \cdot \nabla \hat{I}_0(\mathbf{r}_s, k, p), \end{aligned} \quad (51)$$

where we have defined

$$\hat{T}(\mathbf{r}, k, p) = \hat{S}(\mathbf{r}, k, p) + \hat{A}(\mathbf{r}, k)/c. \quad (52)$$

In deriving equation (50) we have assumed, in an integration by parts, that $I_0(\mathbf{r}, x, t)$ vanishes at $x = \pm\infty$. Now, the source $S(\mathbf{r}, x, t)$, the initial distribution $A(\mathbf{r}, x)$ and the incoming flux $\Gamma(\mathbf{r}_s, x, t)$ all vanish for negative x . Hence, the integral over x defining the Fourier transforms of these quantities need only extend over $(0, \infty)$. Accordingly, the functions $\hat{T}(\mathbf{r}, k, p)$ and $\hat{\Gamma}(\mathbf{r}, k, p)$ are analytic functions of k for $\text{Im } k > 0$, i.e., in the upper half k plane. The solution of equation (50) subject to the boundary condition equation (51) can be symbolized as

$$\begin{aligned} \hat{I}_0(\mathbf{r}, k, p) = [-\nabla \cdot D \nabla + \mu'_a + p/c - ik\mu_s]^{-1} \\ \times \hat{T}(\mathbf{r}, k, p), \end{aligned} \quad (53)$$

where $[]^{-1}$ denotes the inverse of the operator within the brackets. This inverse will exist for all values of k except the eigenvalues of the operator of concern. We therefore consider the location of these eigenvalues in the k plane.

The appropriate eigenvalue problem is

$$-\nabla \cdot D \nabla \psi_j(\mathbf{r}) + (\mu'_a + p/c - ik_j\mu_s)\psi_j(\mathbf{r}) = 0, \quad (54)$$

with boundary condition

$$\frac{1}{4}\psi_j(\mathbf{r}_s) + \frac{1}{2}D\mathbf{n} \cdot \nabla \psi_j(\mathbf{r}_s) = 0, \quad (55)$$

where $\psi_j(\mathbf{r})$ is the j th eigenfunction and k_j is the corresponding eigenvalue. The quantities D , μ'_a , and μ_s are non-negative functions of the spatial variable and $\text{Re } p > 0$. We multiply equation (54) by $\psi_j^*(\mathbf{r})$, where the asterisk indicates the complex conjugate, and integrate over the volume of the system under consideration. We find, using a standard vector identity and Gauss's theorem,

$$\begin{aligned} \int_V d\mathbf{r} D \nabla \psi_j \cdot \nabla \psi_j^* - \int_S ds \psi_j^* D \mathbf{n} \cdot \nabla \psi_j \\ + \int_V d\mathbf{r} (\mu'_a + p/c - ik_j\mu_s)\psi_j \psi_j^* = 0. \end{aligned} \quad (56)$$

Use of equation (55) to eliminate the $\mathbf{n} \cdot \nabla \psi_j$ term in the surface integral and solution of the

result for k_j yields

$$k_j = -i \frac{\int_V d\mathbf{r} [D\nabla\psi_j \cdot \nabla\psi_j^* + (\mu'_a + p/c)\psi_j\psi_j^*] + \frac{1}{2} \int_S d\mathbf{s} \psi_j\psi_j^*}{\int_V d\mathbf{r} \mu_s \psi_j\psi_j^*}. \quad (57)$$

Since $\text{Re } p > 0$, equation (57) shows that $\text{Im } k_j < 0$, i.e. all of the eigenvalues fall in the lower half of the k plane.

Application of the Fourier inversion theorem to equation (53) gives

$$\bar{I}_0(\mathbf{r}, x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} [-\nabla \cdot D\nabla + \mu'_a + p/c - ik\mu_s]^{-1} \hat{T}(\mathbf{r}, k, p). \quad (58)$$

For negative x we evaluate equation (58) by closing the contour in the upper half plane. The semicircular contour gives a zero contribution because of the e^{-ikx} term in the integrand. Further, we have shown that $\hat{T}(\mathbf{r}, k, p)$ and the inverse operator are analytic in the upper half of the k plane. By Cauchy's theorem we then conclude

$$\bar{I}_0(\mathbf{r}, x, p) = 0, x < 0. \quad (59)$$

Inversion of the Laplace transform in equation (59) yields

$$I_0(\mathbf{r}, x, t) = \frac{1}{2\pi i} \int dp e^{pt} \bar{I}_0(\mathbf{r}, x, p) = 0, x < 0,$$

thus completing the proof that the intensity vanishes for negative x .

We now consider a similar analysis for the transport equation, (23). The equation we shall analyze, analogous to equation (46) for the

diffusion approximation, is

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\mathbf{r}, x, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{r}, x, \boldsymbol{\Omega}, t) \\ & + \mu'_a I(\mathbf{r}, x, \boldsymbol{\Omega}, t) = -\mu_s I(\mathbf{r}, x, \boldsymbol{\Omega}, t) \\ & + \frac{3\mu_s}{16\pi} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] I(\mathbf{r}, x, \boldsymbol{\Omega}', t) \\ & - \frac{\mu_s}{4\pi} \int_{4\pi} d\boldsymbol{\Omega}' \frac{\partial I(\mathbf{r}, x, \boldsymbol{\Omega}', t)}{\partial x} \\ & + S(\mathbf{r}, x, \boldsymbol{\Omega}, t) H(x), \quad -\infty < x < \infty, \quad (60) \end{aligned}$$

with initial and boundary conditions

$$I(\mathbf{r}, x, \boldsymbol{\Omega}, 0) = A(\mathbf{r}, x, \boldsymbol{\Omega}) H(x), \quad (61)$$

$$I(\mathbf{r}_s, x, \boldsymbol{\Omega}, t) = \Gamma(\mathbf{r}_s, x, \boldsymbol{\Omega}, t) H(x), \boldsymbol{\Omega} \cdot \mathbf{n} < 0, \quad (62)$$

where μ'_a and μ_s are functions of the spatial variable only. As in the diffusion case, we wish to prove that $I(\mathbf{r}, x, \boldsymbol{\Omega}, t)$ is identically zero for negative x . Application of the Laplace transform with respect to t and the Fourier transform with respect to x yields

$$\begin{aligned} & \frac{p}{c} \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}, p) + \boldsymbol{\Omega} \cdot \nabla \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}, p) \\ & + \mu'_a \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}, p) = -\mu_s \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}, p) \\ & + \frac{3\mu_s}{16\pi} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}', p) \\ & + ik\mu_s \int_{4\pi} d\boldsymbol{\Omega}' \hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}', p) + \hat{T}(\mathbf{r}, k, \boldsymbol{\Omega}, p), \quad (63) \end{aligned}$$

$$\hat{I}(\mathbf{r}_s, k, \boldsymbol{\Omega}, p) = \hat{\Gamma}(\mathbf{r}_s, k, \boldsymbol{\Omega}, p), \boldsymbol{\Omega} \cdot \mathbf{n} < 0, \quad (64)$$

where we have defined

$$\hat{T}(\mathbf{r}, k, \boldsymbol{\Omega}, p) = \hat{S}(\mathbf{r}, k, \boldsymbol{\Omega}, p) + \hat{A}(\mathbf{r}, k, \boldsymbol{\Omega})/c. \quad (65)$$

The solution of equations (63) and (64) can be symbolized

$$\hat{I}(\mathbf{r}, k, \boldsymbol{\Omega}, p) = O^{-1} \hat{T}(\mathbf{r}, k, \boldsymbol{\Omega}, p), \quad (66)$$

where the operator O^{-1} is the inverse of the operator of equation (63). Application of the inverse Fourier and Laplace transformation

gives the intensity as

$$I(\mathbf{r}, x, \boldsymbol{\Omega}, t) = \frac{1}{4\pi^2 i} \int dp e^{pt} \int_{-\infty}^{\infty} dk e^{-ikx} O^{-1} \hat{T}(\mathbf{r}, k, \boldsymbol{\Omega}, p). \quad (67)$$

If the integrand of equation (67) is an analytic function of k in the upper half plane, closure of the contour, for negative x , in the upper half of the k plane shows that

$$I(\mathbf{r}, x, \boldsymbol{\Omega}, t) = 0, \quad x < 0, \quad (68)$$

as was to be proved. Since the source, initial distribution, and incoming flux distribution vanish for negative x , it is known that the transforms $\hat{T}(\mathbf{r}, k, \boldsymbol{\Omega}, p)$ and $\hat{\bar{T}}(\mathbf{r}, k, \boldsymbol{\Omega}, p)$ are analytic in the upper half of the k plane. Hence to show that the integrand in equation (67) has the proper analyticity properties, we need only show that the eigenvalues k_j of the operator O lie in the lower half of the k plane.

The eigenvalue problem to be studied is

$$\begin{aligned} \boldsymbol{\Omega} \cdot \nabla \psi_j(\mathbf{r}, \boldsymbol{\Omega}) + (\mu'_a + \mu_s + p/c) \psi_j(\mathbf{r}, \boldsymbol{\Omega}) \\ = \frac{3\mu_s}{16\pi} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] \psi_j(\mathbf{r}, \boldsymbol{\Omega}') \\ + ik_j \mu_s \int_{4\pi} d\boldsymbol{\Omega}' \psi_j(\mathbf{r}, \boldsymbol{\Omega}'), \end{aligned} \quad (69)$$

with boundary condition

$$\psi_j(\mathbf{r}, \boldsymbol{\Omega}) = 0, \quad \mathbf{n} \cdot \boldsymbol{\Omega} < 0. \quad (70)$$

Here $\psi_j(\mathbf{r}, \boldsymbol{\Omega})$ is the j th eigenfunction, k_j is the corresponding eigenvalue, the quantities μ'_a and μ_s are non-negative functions of space, and $\text{Re } p > 0$. We multiply equation (69) by $\psi_j^*(\mathbf{r}, \boldsymbol{\Omega})$, where the asterisk denotes the complex conjugate, and integrate over the volume of the system and over all solid angles. If we add to this result its own complex conjugate and apply Gauss's theorem, we find

$$\begin{aligned} \frac{1}{2} \int_S ds \int_{4\pi} d\boldsymbol{\Omega} \mathbf{n} \cdot \boldsymbol{\Omega} \psi_j \psi_j^* + \int_V d\mathbf{r} (\mu'_a + p_R/c) \int_{4\pi} d\boldsymbol{\Omega} \psi_j \psi_j^* + \int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \psi_j \psi_j^* - \frac{3}{16\pi} \int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] \psi_j(\boldsymbol{\Omega}') \psi_j^*(\boldsymbol{\Omega}) \\ = -k_{JI} \int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \int_{4\pi} d\boldsymbol{\Omega}' \psi_j(\boldsymbol{\Omega}') \psi_j^*(\boldsymbol{\Omega}), \end{aligned} \quad (71)$$

where the subscripts R and I indicate real and imaginary parts, respectively. Using the bound-

ary condition, equation (70), in equation (71) and solving for k_{JI} we find

$$\begin{aligned} k_{JI} = - \frac{\frac{1}{2} \int_S ds \int_{\mathbf{n} \cdot \boldsymbol{\Omega} > 0} d\boldsymbol{\Omega} \mathbf{n} \cdot \boldsymbol{\Omega} \psi_j \psi_j^* + \int_V d\mathbf{r} (\mu'_a + p_R/c) \int_{4\pi} d\boldsymbol{\Omega} \psi_j \psi_j^*}{\int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \int_{4\pi} d\boldsymbol{\Omega}' \psi_j(\boldsymbol{\Omega}') \psi_j^*(\boldsymbol{\Omega})} \\ - \frac{\int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \psi_j \psi_j^* - (3/16\pi) \int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \int_{4\pi} d\boldsymbol{\Omega}' [1 + (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}')^2] \psi_j(\boldsymbol{\Omega}') \psi_j^*(\boldsymbol{\Omega})}{\int_V d\mathbf{r} \mu_s \int_{4\pi} d\boldsymbol{\Omega} \int_{4\pi} d\boldsymbol{\Omega}' \psi_j(\boldsymbol{\Omega}') \psi_j^*(\boldsymbol{\Omega})}. \end{aligned} \quad (72)$$

Since $p_R > 0$, equation (72) shows that all of the eigenvalues in question fall in the lower half of the k plane if we can show that the numerator of the second term is non-negative. Since μ_s is a non-negative function of \mathbf{r} , it suffices to show

$$\int_{4\pi} d\Omega g(\Omega) g^*(\Omega) - \frac{3}{16\pi} \int_{4\pi} d\Omega \int_{4\pi} d\Omega' \times [1 + (\Omega \cdot \Omega')^2] g(\Omega) g^*(\Omega') \geq 0 \quad (73)$$

for an arbitrary function $g(\Omega)$.

We shall prove the more general result that

$$\int_{4\pi} d\Omega g(\Omega) g^*(\Omega) - \int_{4\pi} d\Omega \int_{4\pi} d\Omega' f(\Omega \cdot \Omega') g(\Omega) g^*(\Omega') \geq 0 \quad (74)$$

for an arbitrary function $g(\Omega)$ and any real non-negative function $f(\Omega \cdot \Omega')$ with the property

$$\int_{4\pi} d\Omega f(\Omega \cdot \Omega') = 2\pi \int_{-1}^1 d\mu f(\mu) = 1. \quad (75)$$

We expand $f(\Omega \cdot \Omega')$ in Legendre polynomials according to

$$f(\Omega \cdot \Omega') = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} f_n P_n(\Omega \cdot \Omega'), \quad (76)$$

where

$$f_n = \int_{4\pi} d\Omega P_n(\Omega \cdot \Omega') f(\Omega \cdot \Omega') = 2\pi \int_{-1}^1 d\mu P_n(\mu) f(\mu). \quad (77)$$

We further expand $g(\Omega)$ in the surface harmonics $Y_{nm}(\Omega)$, defined by equation (18), as

$$g(\Omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm} g_{nm} Y_{nm}(\Omega), \quad (78)$$

where the A_{nm} are defined by equation (19) and

$$g_{nm} = \int_{4\pi} d\Omega Y_{nm}^*(\Omega) g(\Omega), \quad (79)$$

with the asterisk indicating the complex conjugate. Use of equations (76) and (78) in the left hand side of equation (74) together with the

addition formula [17]

$$P_n(\Omega \cdot \Omega') = \sum_{m=-n}^n \frac{(n-|m|)!}{(n+|m|)!} Y_{nm}(\Omega) Y_{nm}^*(\Omega') \quad (80)$$

and the biorthogonality relationship between the surface harmonics and their complex conjugates yields

$$\int_{4\pi} d\Omega g(\Omega) g^*(\Omega) - \int_{4\pi} d\Omega \int_{4\pi} d\Omega' f(\Omega \cdot \Omega') g(\Omega) g^*(\Omega') = \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm} (1 - f_n) g_{nm} g_{nm}^*. \quad (81)$$

The left-hand side of equation (81) will be non-negative if we can show that $f_n \leq 1$ for all n . Recalling that $f(\Omega \cdot \Omega')$ is a non-negative function with a normalization given by equation (75), we have, using the definition of f_n , equation (77),

$$f_n = \frac{\int_{-1}^1 d\mu P_n(\mu) f(\mu)}{\int_{-1}^1 d\mu f(\mu)} \leq \frac{\int_{-1}^1 d\mu |P_n(\mu)| f(\mu)}{\int_{-1}^1 d\mu f(\mu)} \leq 1, \quad (82)$$

the last inequality following from the fact that the maximum value of $|P_n(\mu)|$ in the range $(-1, 1)$ is unity. This completes the proof that the imaginary part of all the eigenvalues k_j for the transport operator is negative, which in turn shows that the intensity $I(\mathbf{r}, x, \Omega, t)$ vanishes for negative x .

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Résumé—L'équation de transport appropriée pour les calculs de transport de chaleur par rayonnement et tenant compte des effets de dispersion des photons est discutée. On montre que pour des températures et des énergies de photon faibles ($\ll 0,5$ MeV), la dispersion des photons par un gaz maxwellien d'électrons libres peut être décrite par un opérateur différentiel du second ordre relativement simple. Une approximation du type de la diffusion, employant la même description de la dispersion, est obtenue à partir de l'équation de transport. Les aspects qualitatifs de la solution des équations de transport et de diffusion sont discutés dans le cas limite d'une température électronique nulle. En particulier, on montre que dans cette limite cette description de la dispersion donne le comportement propre seulement d'une décroissance de la fréquence des photons au course de leur dispersion. Ceci contraste avec des essais antérieurs pour représenter le processus de dispersion par un opérateur différentiel qui conduisait au résultat physiquement incorrect d'une certaine croissance de la fréquence due à la dispersion à partir des électrons au repos.

Zusammenfassung—Die für die Berechnung des Wärmeübergangs durch Strahlung geeignete Transportgleichung, welche Effekte der Photonenstreuung einschliesst, wird diskutiert. Es wird gezeigt, dass für kleine Temperaturen und Photonenenergien ($\ll 0,5$ MEV) die Streuung der Photonen eines Maxwell'schen Gases freier Elektronen durch einen relative einfachen Differentialoperator zweiter Ordnung beschrieben werden kann. Eine Näherung für die Diffusion, wird aus der Transportgleichung abgeleitet, wobei dieselbe Beschreibung der Streuung verwendet wird. Die qualitativen Aspekte der Lösung, sowohl der Transportals auch der Diffusionsgleichung werden für den Grenzfall der Elektronentemperatur null diskutiert. Insbesondere wird gezeigt, dass in diesem Grenzfall diese Beschreibung der Streuung das richtige Verhalten bei der Streuung, nämlich nur eine Abnahme der Photonenfrequenz, angibt.

Dies steht im Gegensatz zu früheren Versuchen, die den Prozess der Streuung durch einen Differentialoperator ausdrücken, wobei sich das unkorrekte Ergebnis eines Anwachsens der Frequenz ergibt, als Folge der Streuung an ruhenden Elektronen.

Аннотация—Рассматривается уравнение лучистого теплообмена с учётом фотонного рассеяния. Показано, что при низких температурах и энергиях фотонов ($\ll 0,5$ мэв) рассеяние фотонов максвелловским газом свободных электронов можно описать с помощью относительно простого дифференциального оператора второго порядка. Из уравнения переноса выведена диффузионная аппроксимация, также учитывающая рассеяние. Количественные результаты решения уравнений переноса и диффузии рассматриваются в предельном случае нулевой электронной температуры. При этом оказалось, что рассеяние приводит только к уменьшению частоты излучения. Прежние попытки представить процесс рассеяния с помощью дифференциального оператора давали физически неверные результаты некоторого увеличения частоты за счет рассеяния.